TOPOLOGY - III, EXERCISE SHEET 9

Exercise 1. *Naturality of the Hurewicz homomorphism.*
Recall from the lectures that given a path-connected space X, there exists a well defined group homomorphism $h : \pi_n(X) \to H_n(X)$ for all $n \geq 1$. Let u be a fixed generator of $H_n(S^n)$, then the homomorphism h is given by the assignment $f \mapsto f_*(u)$. Show that the Hurewicz homomorphism is functorial in X. That is given a continuous map $\phi : X \to Y$ of topological spaces, show that the following diagram commutes:

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{\phi_*} & \pi_n(Y) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ H_n(X) & \xrightarrow{\phi_*} & H_n(Y). \end{array}$$

Where we abuse notation by denoting both the horizontal pushforward maps on homotopy groups and homology groups as ϕ_* and both the vertical Hurewicz maps as h. Note that in the language of category theory the above exercise says that h is a natural transformation between the functors π_n and H_n.

Exercise 2. *H_1 of a surface via Hurewicz Theorem.*
Let X be a path connected space. Recall from the lectures that the Hurewicz homomorphism induces an isomorphism of abelian groups:

$$\pi_1^{ab}(X) \xrightarrow[h]{\sim} H_1(X).$$

Where $\pi_1(X)^{ab}$ denotes the abelianisation of the fundamental group of X. The goal of this exercise is to compute the first homology group of all compact connected surfaces X.

(1) Recall van Kampen's theorem that if a path connected space X is a union of open subspaces U_1, U_2 which are themselves path connected and are such that the intersection $U_1 \cap U_2$ is non-empty and path connected then the following diagram is a pushout diagram.

$$\begin{array}{ccccc} & & \pi_1(U_1) & \xrightarrow{\quad j_1 \quad} & \\ & \nearrow^{i_1} & & \searrow & \\ \pi_1(U_1 \cap U_2) & & & & \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \overset{\cong}{\dashrightarrow} \pi_1(X) \, . \\ & \searrow_{i_2} & & \nearrow & \\ & & \pi_1(U_2) & \xrightarrow{\quad j_2 \quad} & \end{array}$$

In other words $\pi_1(X)$ is characterised uniquely up to isomorphism by the property that a homomorphism of groups from $\pi_1(X)$ to a group G is the same as a pair of homomorphisms $\phi_1 : \pi_1(U_1) \to G$ and $\phi_2 : \pi_1(U_2) \to G$ such that $\phi_1 \circ i_1 = \phi_1 \circ i_2$.

Show that if U_2 is contractible then $\pi_1(X) \cong \pi_1(U_1)/\langle i_1(\pi_1(U_1 \cap U_2)\rangle$. Where $\langle i_1(\pi_1(U_1 \cap U_2)\rangle$ is the normal subgroup generated by $i_1(\pi_1(U_1 \cap U_2)$.

(2) Recall from exercise 5 of sheet 3 that every surface Σ has a standard planar model using which it can be expressed as a quotient of a polygon. The standard planar model for $(T^2)^{\#n}$ is given by $a_1 b_1 a_1^{-1} b_1^{-1} ... a_n b_n a_n^{-1} b_n^{-1}$ and that for $(\mathbb{RP}^2)^{\#n}$ is given by $a_1 a_1 a_2 a_2 ... a_n a_n$. Using van Kampen's theorem show that

$$\pi_1((T^2)^{\#n}) = \langle\ a_1, b_1, ..., a_n, b_n \mid [a_1, b_1] \cdot \ldots \cdot [a_n, b_n] = 1\ \rangle.$$

and

$$\pi_1((\mathbb{RP}^2)^{\#n}) = \langle\ a_1, ..., a_n \mid a_1^2 \cdot a_2^2 \ldots \cdot a_n^2 = 1\ \rangle.$$

Hint: To use van Kampen's theorem come up with an open cover of the surface Σ in question by creating a cover of the planar model. Let U_1 be the open subset of the planar model given by a puncture at the centre and U_2 be the open given by a small open disk around the centre.

(3) Use the Hurewicz theorem to compute $H_1(\Sigma)$ for all compact connected surfaces Σ. That is compute the abelianisations of the fundamental groups in part (2).